Journal of Statistical Physics, Vol. 16, No. 1, 1977

Long-Time Behavior of the Nonlocal Shear Viscosity of a One-Component Plasma: A Microscopic Approach

M. Baus[1,2] and J. Wallenborn[1]

Received May 24, 1976

The nonlocal shear viscosity $\eta(t)$ of a classical one-component plasma is shown to have an *oscillatory* long-time tail. This result is obtained on the basis of a *microscopic* theory which does not rely on expansions in a small parameter such as the plasma expansion parameter. Our major approximation is the restriction to the coupling of *two* hydrodynamic propagators in the computation of the long-time behavior of the transport matrix. The Coulomb divergence is correctly accounted for, while the nonanalyticities of both the plasma parameter and gradient expansions are discussed at the level of the kinetic as well as the hydrodynamic equations.

KEY WORDS: Plasma kinetic theory; long-time tails; Coulomb systems; shear viscosity; mode coupling; correlation functions.

1. INTRODUCTION

The discovery by Alder and Wainwright[1] in 1970 of strong computer evidence for the existence of a nonexponential decay of the velocity correlation function of a system of hard spheres has led to one of the most spectacular developments of recent years in the kinetic theory of neutral particles. All these developments are based on the recognition of the importance of *hydrodynamic transport of fluctuations* for the study of phenomena taking place over large space and time intervals. They also provide a coherent view of such related results as the nonexistence of naive density expansions, the nonexponential decay of the nonlocal transport coefficients, and, last but not least, the nonexistence of hydrodynamic equations at Burnett or super-Burnett order. Similar results have been obtained for a variety of systems of various dimensionality by a variety of methods, ranging from purely

[1] Chimie Physique II, Université Libre de Bruxelles, Bruxelles, Belgium.
[2] Chercheur Qualifié du Fonds National Belge de la Recherche Scientifique.

phenomenological to "exact" kinetic theoretical ones. An extensive list of references can be found in a recent review article by Pomeau and Résibois,[2] to which we will frequently refer.

1.1. The One-Component Plasma

In this paper we turn our attention to the classical plasma, a system which remained outside the scope of the previous developments, which were restricted to the case of short-range forces. In the presence of Coulomb forces a number of basic results from the kinetic theory of neutral particles have to be reconsidered, as Baus has already pointed out.[3-5] As the simplest of such Coulomb systems, we will consider here the classical one-component plasma, i.e., a one-species system of charged point particles with pure Coulomb interactions embedded in an inert background of opposite charge ensuring overall electroneutrality. Very fortunately, an impressive amount of results from molecular dynamics for this system has been obtained recently by Hansen *et al.*[6-10] Some of these results can be explained on the basis of the theory presented below, while others still wait for a microscopic explanation and appear to be very challenging.

The scope of the theory to be constructed is dictated by two preliminary observations. First, all the phenomena we have in view manifest themselves in the hydrodynamic correlation functions, i.e., the space–time equilibrium correlation functions of the conserved densities, such as the number density, the momentum density, and the energy density. Next, these phenomena do not depend on the smallness of the density n, coupling constant e^2, or plasma expansion parameter $\lambda = k_D{}^3/n$ [here $k_D = (4\pi e^2 n\beta)^{1/2}$ is the Debye wave vector of a one-component plasma of particles of charge e, number density n, and equilibrium temperature T, and $\beta = (\kappa T)^{-1}$, with κ Boltzmann's constant]. What we really need is thus a fairly general theory for the hydrodynamic correlation functions of a one-component plasma.

1.2. The Coulomb Divergence, the Gradient Expansion, and the Local Equilibrium Problem

One is immediately tempted to gain some information about these correlation functions from the Landau–Placzek theory, which proved extremely useful in the neutral particle case.[11,12] However, the local equilibrium distribution as well as the hydrodynamic equations used in these theories emerged from a small gradient expansion of some more basic kinetic equation, and such a small gradient expansion of the kinetic equation does not exist in the plasma case. Indeed, an essential ingredient of any kinetic equation describing a plasma is the so-called Vlasov term $e\mathbf{E}(\mathbf{r}, t)\cdot$

$(\partial/\partial\mathbf{p})f(\mathbf{r}, \mathbf{p}, t)$, where $\mathbf{E}(\mathbf{r}, t)$ is the self-consistent electric field at point $\mathbf{r}$ and time t, and f is the one-particle distribution of particles of momentum $\mathbf{p}$. Omitting for the moment unnecessary features, we can write

$$\mathbf{E}(\mathbf{r}) \sim \nabla \int d\mathbf{r}' \, V(\mathbf{r} - \mathbf{r}')f(\mathbf{r}')$$
$$= \int d\mathbf{r}' \, V(\mathbf{r}') \, \nabla f(\mathbf{r} - \mathbf{r}') = \int d\mathbf{r}' \, V(\mathbf{r}') \, \nabla f(\mathbf{r}) + O(\nabla^2)$$

When $V(\mathbf{r})$ is the Coulomb potential, the integral $\int d\mathbf{r}' \, V(\mathbf{r}')$ does not exist and the gradient expansion breaks down. This rough argument can be turned around or refined, e.g., the potential can be renormalized into the Ornstein–Zernike direct correlation function, but the Coulomb divergence will persist.[4] It simply expresses the basic fact that in the presence of the long-range Coulomb forces the interaction is basically *nonlocal*. The zeroth-order, *local equilibrium* distribution cannot be attained and the standard Chapman–Enskog derivation of the hydrodynamic equations fails. Does this mean that Landau–Placzek theories are completely useless in the plasma context? We think there are at least two situations where they can give reasonable indications. First, if we *restrict* ourselves to disturbances or fluctuations which do not disturb the electroneutrality over appreciable distances, then the Vlasov term, whose Fourier transform is proportional to the charge density, can be dominated by the collision term and the local equilibrium distribution should yield a reasonable first approximation. In fact, although usually not stated explicitly, it is this situation which is termed plasma hydrodynamics in the literature.[13] Such a subsidiary condition on the type of disturbances one considers can be realized in a two-component plasma (however, are they maintained in time?), but not in a one-component plasma because here the density fluctuation is always proportional to the charge fluctuation. Next, for the local equilibrium state to represent a reasonable approximation for a one-component plasma one should consider small but finite wave vectors ($\sim$ gradients) and consider situations where the plasma expansion parameter is large enough for the collision term still to dominate the Vlasov term. However, such an accumulation of subsidiary conditions leads us, in our opinion, too far away from what linearized hydrodynamics really is, namely a theory which, after a transient time, becomes asymptotically exact for vanishingly small wave vectors for any system. For neutral particle systems detailed proofs of this statement are now available in the literature.[14,15] For the one-component plasma a similar result has been published recently by Baus,[4] while a separate proof as needed for a two-component plasma is now underway.[5] So, in order to cope with the fairly general situations we have in mind, we will develop the microscopic approach of Ref. 4 within the present context; as we will show, this can be done quite simply.

In Section 2 we summarize our previous results. The developments necessary to tackle the problem of the long-time behavior of the nonlocal transport coefficients of a one-component plasma are given in Section 3 and are applied to the case of the shear viscosity in Section 4. The major implications of the results obtained are reviewed in Section 5.

2. KINETIC THEORY FOR THE CORRELATION FUNCTIONS

We now summarize some of the results on the hydrodynamic correlation functions obtained elsewhere.[4] As they will be used extensively below, we urge the reader to consult Ref. 4 for the details.

2.1. Two-Point Correlation Function

Our starting point will be the following *exact* kinetic equation obeyed by the two-point correlation function $S(1, 2; t - t') = \langle \delta f(1, t)\, \delta f(2, t')\rangle$, $\alpha \equiv (\mathbf{r}_\alpha, \mathbf{p}_\alpha)$, $\alpha = 1, 2$, of the equilibrium fluctuations $\delta f = f - \langle f\rangle$ of the phase-space density $f(1, t) = \sum_{j=1}^{N} \delta(\mathbf{r}_1 - \mathbf{x}_j(t))\, \delta(\mathbf{p}_1 - \mathbf{p}_j(t))$:

$$zS(\mathbf{k}z; \mathbf{p}_1\mathbf{p}_2) - \int d\mathbf{p}_3\, \Sigma(\mathbf{k}z; \mathbf{p}_1\mathbf{p}_3)S(\mathbf{k}z; \mathbf{p}_3\mathbf{p}_2) = iS(\mathbf{k}, t = 0; \mathbf{p}_1\mathbf{p}_2) \quad (1)$$

where $S(\mathbf{k}z; \mathbf{p}_1\mathbf{p}_2)$ is the Fourier–Laplace transform of $S(1, 2; t - t')$:

$$\begin{aligned} S(\mathbf{k}z; \mathbf{p}_1\mathbf{p}_2) &= \int d(\mathbf{r}_1 - \mathbf{r}_2) \int_0^\infty d(t - t') \\ &\quad \times \{\exp[-i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2) + iz(t - t')]\}S(1, 2; t - t'); \qquad \operatorname{Im} z > 0 \end{aligned} \quad (2)$$

The basic kinetic equation (1) can be obtained directly from the Liouville equation obeyed by $f(1, t)$ through a few algebraic steps originally developed by Akcasu and Duderstadt as a straightforward application of Mori's method to $f(1, t)$. Explicit derivations of Eq. (1) can be found in the literature.[16,14,4] All known *linearized* kinetic equations can be obtained as particular cases of (1). The memory function Σ in (1) can be split into three distinct parts, $\Sigma = \Sigma^0 + \Sigma^s + \Sigma^c$, a free-streaming term $\Sigma^0(\mathbf{k}; \mathbf{p}_1\mathbf{p}_2) = \mathbf{k}\cdot\mathbf{v}_1\, \delta(\mathbf{p}_1 - \mathbf{p}_2)$ with $\mathbf{p}_i = m\mathbf{v}_i$, a static self-consistent field term $\Sigma^s(\mathbf{k}; \mathbf{p}_1) = -\mathbf{k}\cdot\mathbf{v}_1 c(\mathbf{k})\varphi(\mathbf{p}_1)$, and a nonlocal (linearized) collision operator $\Sigma^c(\mathbf{k}z; \mathbf{p}_1\mathbf{p}_2)$ whose explicit expression has been given in Ref. 4. Here $\varphi(\mathbf{p}) = (\beta/2\pi m)^{3/2} \exp(-\beta\mathbf{p}^2/2m)$ is the Maxwellian and $c(\mathbf{k})$ is the Ornstein–Zernike direct correlation function. At $\mathbf{k} = 0$, $c(\mathbf{k})$ is singular[4] because as $k \to 0$, $c(k) \sim O(k^{-2})$ and consequently the small-k expansion of the kinetic equation (1) does not exist, as already stated in the introduction. To avoid this difficulty we *first* project Eq. (1) onto the subspace spanned by the hydrodynamic moments, which can be considered to be the five first members of a momentum space basis $\langle i|$, $i = 1,\ldots, 5$.

2.2. Hydrodynamic Correlation Functions

The kinetic equation (1) is now transformed into a finite set of algebraic equations for the hydrodynamic correlation functions $G_{ij}(\mathbf{k}z) = \langle i|S(n\varphi)^{-1}|j\rangle$

$$\sum_{j=1}^{5} [z\,\delta_{ij} - \Omega_{ij}(\mathbf{k}z)]G_{ji'}(\mathbf{k}z) = iG^0_{ii'}(k); \qquad (i, i') = (1,\ldots, 5) \tag{3}$$

Here $G^0_{ij}(k) = \langle i|S(\mathbf{k}, t = 0)(n\varphi)^{-1}|j\rangle$ are static correlation functions, which can easily be obtained[4] from the *known* initial condition $S(\mathbf{k}, t = 0; \mathbf{p}_1\mathbf{p}_2)$ appearing in the rhs of (1). Solving the algebraic equations (3), we obtain *exact* expressions for the hydrodynamic correlation functions $G_{ij}(\mathbf{k}z)$ in terms of $G^0_{ij}(k)$ and the transport matrix, whose expression is given below in (11). The contributions to $G_{ij}(\mathbf{k}, t)$ can be split unambiguously as $G_{ij}(\mathbf{k}, t) = G^H_{ij}(\mathbf{k}, t) + \bar{G}_{ij}(\mathbf{k}, t)$, where $\bar{G}_{ij}(\mathbf{k}, t)$ denotes the contributions to $G_{ij}(\mathbf{k}, t)$ that originate from the roots $z = z(k)$ of the dispersion relation, $\det|z - \Omega(kz)| = 0$, whose imaginary part (damping rate) remains finite as k goes to zero (relaxation modes), while $G^H_{ij}(\mathbf{k}t)$ originates from the so-called hydrodynamic modes, which have a damping rate of $O(k^2)$ and hence vanish with k. For small k, G_{ij} can be approximated by G^H_{ij} after a transient time of $O(k^{-2})$. Moreover, for small k, $G^H_{ij}(\mathbf{k}t)$ can be computed exactly even in the Coulomb case, because, as shown elsewhere,[4] the dispersion relation admits, contrary to the kinetic equation (1), a well-defined small-k expansion.

2.3. Hydrodynamic Limit Behavior

Some of the results have already been given in Ref. 4. For later use we complete this list here. When i and j run through the hydrodynamic states (density n, longitudinal momentum l, transverse momentum t_1, t_2, and excess kinetic energy ϵ) we obtain, using the notation of Ref. 4, for $G^H_{ij}(\mathbf{k}, t)$,

$$G^H_{nn}(\mathbf{k}, t) = (k^2/k_D{}^2)\cos[\omega_p(k)t]\exp[-\tfrac{1}{2}k^2\Gamma t] \tag{4}$$

$$G^H_{ll}(\mathbf{k}, t) = \cos[\omega_p(k)t]\exp(-\tfrac{1}{2}k^2\Gamma t) \tag{5}$$

$$G^H_{\epsilon\epsilon}(\mathbf{k}, t) = (c_V{}^0/c_V)\exp(-k^2 D_T t) \tag{6}$$

$$G^H_{nl}(\mathbf{k}, t) = G^H_{ln}(\mathbf{k}, t) = -i(k/k_D)\sin[\omega_p(k)t]\exp(-\tfrac{1}{2}k^2\Gamma t) \tag{7}$$

$$G^H_{l\epsilon}(\mathbf{k}, t) = G^H_{\epsilon l}(\mathbf{k}, t) = \sum_{\pm} \frac{\pm k D_{l\epsilon}(\mathbf{0}, \pm\omega_p)}{2\omega_p[1 - B_\epsilon(\mathbf{0}, \pm\omega_p)]}\exp[-iz_\pm(k)t] \tag{8}$$

$$G^H_{n\epsilon}(\mathbf{k}, t) = G^H_{\epsilon n}(\mathbf{k}, t) = -i\frac{k^2}{k_D{}^2}\frac{c}{v_0}\left(\frac{c_p}{c_V} - 1\right)^{1/2} \exp(-k^2 D_T t)$$
$$+ \sum_{\pm} \frac{i}{2}\frac{k^2}{k_D{}^2}\frac{D_{\epsilon l}(\mathbf{0}, \pm\omega_p)}{v_0[1 - B_\epsilon(\mathbf{0}, \pm\omega_p)]}$$
$$\times \exp[-iz_\pm(k)t] \tag{9}$$

and

$$G_{t_\alpha j}(\mathbf{k}, t) = G_{jt_\alpha}(\mathbf{k}, t) = \delta_{j,t_\alpha} G_\perp(\mathbf{k}, t) \tag{10a}$$

$$G^H_\perp(\mathbf{k}, t) = \exp(-k^2 D_\perp t) \tag{10b}$$

where in order to avoid confusion with the time variable we have slightly departed from the notation used in Ref. 4: $G_\perp \equiv G_{tt}$ and $D_\perp \equiv D_t$. In Eqs. (4)–(10) we only wrote down, as one should, the dominant contribution of the small-k expansion. Let us now briefly recall the physical contents of Eqs. (4)–(10). The time dependence of the hydrodynamic correlation functions $G^H_{ij}(\mathbf{k}, t)$ is completely determined by the five hydrodynamic or long-wavelength modes of our system: two transverse shear modes $z_{t_\alpha} = -ik^2 D_\perp$ ($\alpha = 1, 2$), one thermal mode $z_T = -ik^2 D_T$, and two plasma modes $z_\pm(k) = \pm\omega_p(k) - \frac{1}{2}ik^2\Gamma$, $\omega_p(k) = \omega_p(1 + \frac{1}{2}k^2\gamma)$. No sound modes appear in the one-component plasma because the density fluctuations are identical to the charge density fluctuations of this system, which themselves are controlled by Coulomb phenomena. The high-frequency plasma modes are only weakly coupled to the low-frequency thermal mode, and hence the thermal diffusitivity D_T differs by a factor c_p/c_V, the specific heat ratio, from its neutral fluid analog. The (collisional) damping rate Γ and the dispersion coefficient of the plasma frequency γ require a knowledge of the non-Markovian or finite-frequency collision operator and hence cannot be expressed in terms of (zero-frequency) transport coefficients. This clearly is a supplementary reason why, as stated in the introduction, local-equilibrium concepts cannot be used here. This departure from the usual (neutral fluid) hydrodynamic behavior is even reinforced if we look at the strengths with which these modes appear in the correlation functions (4)–(10). We observe, for example, that (4), (5), and (7) are completely dominated by the plasma modes, while (6) is dominated by the thermal mode. The strength of the plasma mode in (8) and (9) is non-Markovian, hence nonthermodynamic, in nature. Finally, those correlation functions involving the excess kinetic energy ϵ, and not the total energy, have an initial value $G^H_{ij}(\mathbf{k}, t = 0)$ which differs from the exact one $G_{ij}(\mathbf{k}, t = 0)$ because part of the kinetic energy fluctuations are not properly described by a hydrodynamic theory.[17,4]

As they only involve the small-k limit, the above results can be termed "exact."

The molecular dynamics results of Hansen *et al.*[6–10] do contain information about $G_\perp$ of Eq. (10b) and about the related functions G_{nn}, G_{ll}, and G_{nl} of Eqs. (4), (5), and (7). Although no detailed comparison has been undertaken, we agree on the most striking features[8,10]: (1) the absence of a thermal Rayleigh peak in G_{nn}^H and (2) the failure of a mean-field theory as well as a Landau–Placzek theory to reproduce correctly the location $[\omega_p(k)]$ and width $(k^2\Gamma)$ of the plasmon peaks, especially as one varies the plasma expansion parameter.[28]

3. KINETIC THEORY OF THE LONG-TIME BEHAVIOR OF THE TRANSPORT MATRIX

The long-wavelength modes that build up the hydrodynamic correlation functions $G_{ij}^H(\mathbf{k}, t)$ of Eqs. (4)–(10) have been given detailed expression in Ref. 4 in terms of the transport matrix $\Omega_{ij}(\mathbf{k}z)$ of Eq. (3). This nomenclature is based on the fact that the transport coefficients which appear in these modes are obtained as limits of $\Omega_{ij}(\mathbf{k}z)$ for vanishing k and z. From the definition of $\Omega_{ij}(\mathbf{k}z)$ we can thus define transport coefficients *nonlocal* in space and time, $\Omega_{ij}(\mathbf{r}t)$, which will appear as integrands in the definition of the usual or *local* transport coefficients. The unexpectedly slow, nonexponential decay of these nonlocal transport coefficients for long times and large space intervals in the case of neutral gases[2] has led us to investigate the same problem for the one-component plasma. In fact the computer results of Hansen *et al.*[8] show that there is a slow nonexponential and oscillatory decay for the nonlocal (in time) self-diffusion coefficient. On the basis of a mode–mode coupling theory[2] one expects a similar behavior for the collective (multiparticle) motions as for the self-motions investigated by Hansen *et al.*[8] Detailed microscopic theories have been obtained independently by Gould and Mazenko for the case of the self-motions, and by the present authors for the case of the collective motions. Brief accounts of both theories have been reported elsewhere.[18,19]

To set up a general microscopic theory of the asymptotic behavior of the nonlocal transport matrix $\Omega_{ij}(\mathbf{r}, t)$ we have to detail somewhat the structure of the nonlocal collision operator $\Sigma^c(\mathbf{k}, z)$ appearing in the kinetic equation (1). Indeed, it is this operator which determines the behavior of the transport matrix Ω_{ij} according to the following relation derived in Ref. 4, whose notation we adopt here:

$$\begin{aligned}\Omega_{ij}(\mathbf{k}z) = \langle i|\bar{\Sigma}(\mathbf{k}z)|j\rangle + \langle i|[\bar{\Sigma}^0(\mathbf{k}) + \bar{\Sigma}^c(\mathbf{k}z)]\bar{Q}\{z - \bar{Q}[\bar{\Sigma}^0(\mathbf{k}) \\ + \bar{\Sigma}^c(\mathbf{k}z)]\bar{Q}\}^{-1}\bar{Q}[\bar{\Sigma}^0(\mathbf{k}) + \bar{\Sigma}^c(\mathbf{k}z)]|j\rangle \qquad (11)\end{aligned}$$

where $\bar{Q} \overset{\circ}{=} I - \bar{\mathbf{P}}$ projects out the hydrodynamic states $(\bar{\mathbf{P}} = \sum_{j=1}^{5} |j\rangle\langle j|)$

and $\bar{\Sigma}$ denotes the operator with elements $\Sigma(pp')$ such that $\langle f|\bar{\Sigma}|g\rangle = \int d\mathbf{p}\, d\mathbf{p}'\, f(\mathbf{p})\Sigma(\mathbf{pp}')\varphi(\mathbf{p}')g(\mathbf{p}')$. According to Eq. (11) the transport matrix Ω_{ij} naturally splits into a direct part and an indirect part, respectively the first and last terms in the rhs of Eq. (11) (we follow here the nomenclature of Ref. 14).

3.1. The Nonlocal Collision Operator

The nonlocal collision operator (or memory function) $\Sigma^c(\mathbf{k}z; \mathbf{pp}')$ can be written in terms of the N-body Liouville operator [see Eq. (2.20d) of Ref. 4] as originally done by Akcasu and Duderstadt,[16,14] or alternatively in terms of a four-point correlation function as done, for example, by Mazenko[20] and Boley.[21] We will follow the latter procedure, in which case we can write[20] Σ^c in terms of two-body dynamics:

$$i\Sigma^c(1, 2; t)n\varphi(2) = \int d1'\, d2'\, L_I(11')L_I(22')C(11', 22'; t) \tag{12}$$

where $1 \equiv (\mathbf{r}_1, \mathbf{p}_1)$, $d1 \equiv d\mathbf{r}_1\, d\mathbf{p}_1$, etc., and $L_I(11')$ is the two-body interaction operator:

$$L_I(11') = -\frac{\partial}{\partial \mathbf{r}_1} V(\mathbf{r}_1 - \mathbf{r}_{1'})\cdot\left(\frac{\partial}{\partial \mathbf{p}_1} - \frac{\partial}{\partial \mathbf{p}_{1'}}\right) \tag{13}$$

for an interaction potential $V(\mathbf{r})$. The four-point correlation function $C(11', 22')$ in Eq. (12) is not the four-point extension, say $S(11', 22')$, of our two-point correlation function $S(12)$ of Eq. (1) but, as shown elsewhere,[20,21] a contracted correlation function. When C is split, by means of a cumulant expansion,[20] into a disconnected part C_D and a connected part C_C, $C = C_D + C_C$, the disconnected part is seen to consist solely of the ordinary two-point correlation functions governed by Eq. (1):

$$C_D(11', 22'; t) = S(12, t)S(1'2', t) + S(12', t)S(1'2, t) \tag{14}$$

We now substitute this separation of C into Eq. (12) and accordingly split the collision operator Σ^c into a disconnected part Σ_D^c and a connected part Σ_C^c: $\Sigma^c = \Sigma_D^c + \Sigma_C^c$. The disconnected contribution to the collision operator can then be written explicitly after integrating by parts the collision term of Eq. (1):

$$\begin{aligned}\Sigma_D^c(\mathbf{k}, t; \mathbf{p}_1\mathbf{p}_2) = -i\int \frac{d\boldsymbol{l}}{8\pi^3}\int d\mathbf{p}_{1'}\, d\mathbf{p}_{2'}\, V_l \boldsymbol{l}\cdot\partial_1 \\ \times [S(\mathbf{k} - \boldsymbol{l}, t; \mathbf{p}_1\mathbf{p}_{2'})S(\boldsymbol{l}, t; \mathbf{p}_{1'}\mathbf{p}_2)V_{l-k}(\boldsymbol{l} - \mathbf{k})\cdot\partial_2 \\ - S(\mathbf{k} - \boldsymbol{l}, t; \mathbf{p}_1\mathbf{p}_2)S(\boldsymbol{l}, t; \mathbf{p}_{1'}\mathbf{p}_{2'})V_l \boldsymbol{l}\cdot\partial_2][n\varphi(2)]^{-1}\end{aligned} \tag{15}$$

where $V_k = 4\pi e^2/k^2$ denotes the Fourier transform of the Coulomb potential (e is the charge on each particle), $\partial_1 \equiv \partial/\partial \mathbf{p}_1$, etc. This is all we need to know about the general structure of the collision operator Σ^c.

3.2. The Basic Approximation

The asymptotic properties of the transport matrix $\Omega_{ij}(\mathbf{k}z)$ are seen from Eq. (11) to be determined by the asymptotic properties of the collision operator $\Sigma^c(\mathbf{k}z)$ (cf. also the discussion of Section 3.4). With respect to this, the separation of the collision operator Σ^c into a disconnected and a connected part, $\Sigma^c = \Sigma_D^c + \Sigma_C^c$, will play an important role. Indeed, comparing (12) and (14), we see that in the disconnected part Σ_D^c the colliding particles propagate through the medium independently of each other with the full two-point propagator $S(\mathbf{k}t; \mathbf{pp}')$. As noted in Section 2, part of this propagation will proceed via the long-lived hydrodynamic fluctuations and hence Σ_D^c will contain a contribution corresponding to the coupling between two *hydrodynamic modes*. In the connected part Σ_C^c we will find instead those contributions to Σ^c in which there are intermediate recollisions between the colliding particles. *If*, as has been shown recently for the neutral gas case,[22] the two-mode coupling contributions dominate the long-time behavior of $\Sigma^c(\mathbf{k}t)$, then we can leave the analysis of the connected part Σ_C^c as such, only assuming it has well-defined limiting values. Moreover, since the connected contributions are *additive* with respect to Σ_D^c, they can, if necessary, always be added in a later stage. Here our basic assumption will then be that *the leading longtime behavior of* $\Sigma^c(\mathbf{k}t)$ *is due solely to the disconnected part* $\Sigma_D^c(\mathbf{k}t)$ displayed in Eq. (15). This is the main approximation in the present theory.

It may be of some interest here to add to the above a remark on the meaning of Σ_D^c. Although we do not need to expand the kinetic equation (1) with respect to the plasma parameter, it could be useful to make some contact between the present theory and the Balescu–Guernsey–Lenard (BGL) kinetic equation, which serves as a reference equation in the kinetic theory of plasmas.[13] This gap is bridged precisely by Σ_D^c. Indeed, if we compute the two-point correlation functions which appear in Σ_D^c [(15)] to zeroth order in the plasma expansion parameter, i.e., use the Vlasov approximation for $S(\mathbf{k}t; \mathbf{pp}')$, and substitute this result for Σ^c into Eq. (1), we obtain a closed kinetic equation. If we consider, moreover, space-independent phenomena ($\mathbf{k} = 0$) and use the Markovian approximation [$\Sigma^c(\mathbf{0}z) \simeq \Sigma^c(\mathbf{00})$] for the collision operator, then this approximate kinetic equation reduces precisely to the linearized BGL equation. One can even go one step further and remove the bare Coulomb potential completely from Eq. (15) in favor of equilibrium correlation functions by using Mazenko's renormalized theory.[20] This will modify the BGL theory considerably at large wave vectors.[6] For instance, the small-distance divergence of the BGL operator can be removed in this way. These interesting questions will be considered in detail elsewhere.[23] Here we are only interested in the long-wavelength behavior, for which the renormalized version of the present theory is not needed.[6] The purpose of the present remark is only to emphasize that $\Sigma_D^c(\mathbf{k}z)$ of Eq. (15) already represents a major extension of the linearized BGL collision operator to finite values of $\mathbf{k}$ and z as well as to finite values of the plasma expansion parameter.

3.3. The Singular Part of the Collision Operator

As stated above, the leading long-time behavior of $\Sigma^c(\mathbf{k}t)$ will come from that part of $\Sigma_D^c(\mathbf{k}t)$, Eq. (15), in which the intermediate propagation proceeds through hydrodynamics. To put in evidence this part of Σ_D^c, we will expand the intermediate propagators of Eq. (15), $S(\boldsymbol{l}t; pp')$, into a complete orthonormalized set of momentum functions $u_i(\mathbf{p})/a_i$ (a_i being a normalization constant), the first five of which are the hydrodynamic states (see Ref. 4). We can then rewrite Eq. (15) as follows:

$$\Sigma_D^c(\mathbf{k}t; \mathbf{p}_1\mathbf{p}_2) = -i\int \frac{d\boldsymbol{l}}{8\pi^3}\sum_{i,j} [A_i(\boldsymbol{l}, 1)G_{in}(\mathbf{k} - \boldsymbol{l}, t)G_{nj}(\boldsymbol{l}, t)\bar{A}_j(\boldsymbol{l} - \mathbf{k}, 2)$$
$$- A_i(\boldsymbol{l}, 1)G_{ij}(\mathbf{k} - \boldsymbol{l}, t)G_{nn}(\boldsymbol{l}, t)\bar{A}_j(\boldsymbol{l}, 2)] \tag{16}$$

where the correlation functions $G_{ij}(\mathbf{k}, t)$ have been defined in Section 2 except that here the state indices i and j run through the infinite set and not just through the first five hydrodynamic states. Furthermore, in Eq. (16) we have put $A_i(\boldsymbol{l}, 1) = a_n V_l \boldsymbol{l}\cdot\partial_1\{[u_i(\mathbf{p}_1)/a_i]n\varphi(1)\}$, where a_n is the normalization constant of the density state[4] ($a_n{}^2 = n$), whereas we can write $\bar{A}_i(\boldsymbol{l}, 1)$ in operator form as $\bar{A}_i(\boldsymbol{l}, 1) = a_n V_l[u_i(1)/a_i]n\varphi(1)\boldsymbol{l}\cdot\partial_1[n\varphi(1)]^{-1}$ (in this case Σ_D^c is an operator with respect to the momentum variable $\mathbf{p}_2$) or, alternatively, integrating by parts the kinetic equation (1), we can write $\bar{A}_i(\boldsymbol{l}, 1) = -[n\varphi(1)]^{-1}A_i(\boldsymbol{l}, 1)$ (in which case Σ_D^c is a function, usually called the memory function). It is of some interest to factor out the $\boldsymbol{l}$-integration in Eq. (16). To do this we introduce a set of three unit vectors $\boldsymbol{\epsilon}_\alpha(\mathbf{k})$ ($\alpha = 1, 2, 3$), such that $\boldsymbol{\epsilon}_\alpha(\mathbf{k})\cdot\boldsymbol{\epsilon}_\beta(\mathbf{k}) = \delta_{\alpha\beta}$ and $\boldsymbol{\epsilon}_3(\mathbf{k}) = \mathbf{k}/|\mathbf{k}|$. We then write Eq. (16) compactly as

$$\Sigma_D^c(\mathbf{k}t; \mathbf{p}_1\mathbf{p}_2) = \sum_{i,j}\sum_{\alpha,\beta} A_i{}^\alpha(\mathbf{p}_1)\Sigma_{ij}^{\alpha\beta}(\mathbf{k}t)\bar{A}_j{}^\beta(\mathbf{p}_2) \tag{17}$$

where the sum over α and β runs over the three polarizations ($\alpha, \beta = 1, 2, 3$), whereas the sum over i and j runs over the infinite set of functions $u_i(p)/a_i$. Furthermore, in Eq. (17) we can put $\bar{A}_j^\beta(\mathbf{p}_2) = -[n\varphi(2)]^{-1}A_j{}^\beta(\mathbf{p}_2)$, with

$$A_j{}^\alpha(\mathbf{p}_1) = a_n\boldsymbol{\epsilon}_\alpha(\mathbf{k})\cdot\partial_1\{[u_j(\mathbf{p}_1)/a_j]n\varphi(\mathbf{p}_1)\} \tag{18}$$

Finally the $\mathbf{k}$, t dependence of Σ_D^c is contained in the following mode coupling integrals:

$$\Sigma_{ij}^{\alpha\beta}(\mathbf{k}t) = -i\int \frac{d\mathbf{l}}{8\pi^3} [\boldsymbol{\epsilon}_\alpha(\mathbf{k})\cdot\mathbf{l}V_l G_{in}(\mathbf{k}-\mathbf{l}, t)G_{nj}(\mathbf{l}, t)\boldsymbol{\epsilon}_\beta(\mathbf{k})\cdot(\mathbf{l}-\mathbf{k})V_{l-k}$$
$$- \boldsymbol{\epsilon}_\alpha(\mathbf{k})\cdot\mathbf{l}V_l G_{ij}(\mathbf{k}-\mathbf{l}, t)G_{nn}(\mathbf{l}, t)\boldsymbol{\epsilon}_\beta(\mathbf{k})\cdot\mathbf{l}V_l] \tag{19}$$

To select now those contributions to Σ_D^c that only contain hydrodynamic propagators we simply have to restrict the sum in (16) or (17) to the hydrodynamic states (i *and* j) $= (1,..., 5)$. These contributions will be shown to be responsible for the slow decay (slower than exponential) of $\Sigma^c(\mathbf{k}t)$ as $t \to \infty$ and $k \to 0$. This then implies that $\Sigma^c(\mathbf{k}z)$ also reaches its limit value $\Sigma^c(\mathbf{k}0)$ slowly as $z \to 0$. Typically we have $\Sigma^c(z) = \Sigma^c(0) + O(z^{1/2})$ if $\Sigma^c(t) \sim O(t^{-3/2})$ as $t \to \infty$. Hence, $\Sigma^c(\mathbf{k}z)$ will not be a regular function of z and $\mathbf{k}$ in the vicinity of $z = 0$ and $\mathbf{k} = 0$. The straightforward expansion of $\Sigma^c(\mathbf{k}z)$ around its local limit $\Sigma^c(\mathbf{00})$ is thus *singular.* Henceforth we will thus split the full collision operator $\Sigma^c(\mathbf{k}z)$ into a regular part Σ_R^c and a singular part Σ_S^c, $\Sigma^c = \Sigma_R^c + \Sigma_S^c$. The singular part is *defined* by the contributions of the hydrodynamic propagators to (17):

$$\Sigma_S^c(\mathbf{k}t; \mathbf{p}_1\mathbf{p}_2) = \sum_{(i,j)=1}^{5} \sum_{(\alpha,\beta)=1}^{3} A_i^\alpha(\mathbf{p}_1)\Sigma_{ij}^{\alpha\beta}(\mathbf{k}t)\bar{A}_j^\beta(\mathbf{p}_2) \tag{20}$$

and it is Σ_S^c that will play the central role in the following analysis.

3.4. The Singular Part of the Transport Matrix

The basic quantity of interest to us is not the collision operator Σ^c but the transport matrix Ω_{ij} of Eq. (11). According to the above, to study the long-time behavior of $\Omega_{ij}(kt)$ we will analyze in more detail the contribution of the singular part of the collision operator Σ_S^c, Eq. (20), to Ω_{ij}. We therefore also separate the transport matrix Ω_{ij} into a regular part Ω_{ij}^R and a singular part Ω_{ij}^S, $\Omega_{ij} = \Omega_{ij}^R + \Omega_{ij}^S$. The regular part Ω_{ij}^R is given by an equation identical to Eq. (11) with Σ replaced by $\Sigma_R = \Sigma - \Sigma_S^c$. The remainder will be called the singular part of the transport matrix:

$$\Omega_{ij}^S(\mathbf{k}z) = \langle i|\bar{\Sigma}_S^c(\mathbf{k}z)|j\rangle + \langle i|\bar{\Sigma}_S^c\bar{Q}(z - \bar{Q}\bar{\Sigma}\bar{Q})^{-1}\bar{Q}\bar{\Sigma}_R|j\rangle$$
$$+ \langle i|\bar{\Sigma}_R\bar{Q}(z - \bar{Q}\bar{\Sigma}\bar{Q})^{-1}\bar{Q}\bar{\Sigma}_S^c|j\rangle$$
$$+ \langle i|\bar{\Sigma}_S^c\bar{Q}(z - \bar{Q}\bar{\Sigma}\bar{Q})^{-1}\bar{Q}\bar{\Sigma}_S^c|j\rangle$$
$$+ \langle i|\bar{\Sigma}_R\bar{Q}(z - \bar{Q}\bar{\Sigma}_R\bar{Q})^{-1}\bar{Q}\bar{\Sigma}_S^c\bar{Q}(z - \bar{Q}\bar{\Sigma}\bar{Q})^{-1}\bar{Q}\bar{\Sigma}_R|j\rangle \tag{21}$$

Next we retain only those contributions to Ω_{ij}^S that are linear in the singular part Σ_S^c, say $\delta\Omega_{ij}$, $\Omega_{ij}^S = \delta\Omega_{ij} + \bar{\Omega}_{ij}^S$, $\bar{\Omega}_{ij}^S$ denoting the remainder. The terms of higher order in Σ_S^c will be dominated by $\delta\Omega_{ij}$ for $z \to 0$, as can be verified a posteriori. Finally, we also introduce a shorthand notation for $\delta\Omega_{ij}$:

$$\delta\Omega_{ij}(\mathbf{k}z) = \langle i|\bar{\Sigma}_S^c(\mathbf{k}z)|j\rangle + \langle i|\bar{\Sigma}_S^c(\mathbf{k}z)|F_j(\mathbf{k}z)\rangle + \langle F_i(\mathbf{k}z)|\bar{\Sigma}_S^c(\mathbf{k}z)|j\rangle + \langle F_i(\mathbf{k}z)|\bar{\Sigma}_S^c(\mathbf{k}z)|F_j(\mathbf{k}z)\rangle \tag{22}$$

where $\langle F_i(\mathbf{k}z)| = \langle i|\bar{\Sigma}_R\bar{Q}(z - \bar{Q}\bar{\Sigma}_R\bar{Q})^{-1}\bar{Q}$, etc. In a few words, $\delta\Omega_{ij}$ represents those contributions to the transport matrix Ω_{ij} that are linear in the singular part of the collision operator $\bar{\Sigma}_S^c$, which in turn represents those contributions to the disconnected collision operator $\bar{\Sigma}_D^c$ [(15), (16)] that contain hydrodynamic propagators [Eq. (20)]. We are now ready to apply the general theory to a specific example.

4. THE NONLOCAL SHEAR VISCOSITY

The theory of Section 3 provides us with a general scheme for the analysis of the feedback action of the hydrodynamic propagators of Section 2 onto themselves. Here we will consider explicitly the case of the shear viscosity, for which we have announced the most important results elsewhere.[19] The other transport phenomena will be considered in future work.

4.1. The Nonlocal Transport Equation

The transverse momentum correlation function $G_\perp(\mathbf{k}z) = G_{t_\alpha t_\alpha}(\mathbf{k}z)$ ($\alpha = 1, 2$) is particularly simple because, for reasons of rotational invariance, $G_\perp$ decouples from the remaining correlation functions and Eq. (3) becomes simply[4]

$$\partial_t G_\perp(\mathbf{k}, t) + i\int_0^t d\tau\, \Omega_\perp(\mathbf{k}, \tau)G_\perp(\mathbf{k}, t - \tau) = 0 \tag{23}$$

with $G_\perp(\mathbf{k}, t = 0) = 1$, where for a change we have transformed Eq. (3) back to the time variable. The transport matrix reduces here to a single element $\Omega_\perp \equiv \Omega_{t_\alpha t_\alpha}$, which because of momentum conservation can be written[4] in terms of a nonlocal shear viscosity $\eta(\mathbf{k}z)$ or shear diffusivity $D_\perp(\mathbf{k}z)$:

$$\Omega_\perp(\mathbf{k}z) = -ik^2 D_\perp(\mathbf{k}z) = -(ik^2/nm)\eta(\mathbf{k}z) \tag{24}$$

where nm is the mass density. As k vanishes, Eq. (23) reduces to

$$\partial_t G_\perp(\mathbf{k}, t) + (k^2/nm)\int_0^t d\tau\, \eta(\mathbf{k} = \mathbf{0}, \tau)G_\perp(\mathbf{k}, t) = 0 \tag{25}$$

and the precise manner in which $\int_0^t d\tau\, \eta(\mathbf{0}\tau)$ reaches its long-time limit $\eta \equiv \int_0^\infty d\tau\, \eta(\mathbf{0}\tau) \equiv \eta(\mathbf{0}, z = 0)$ is the subject of the present section. In this limit Eq. (25) reduces to the transverse Navier–Stokes equation $\partial_t G_\perp + (k^2\eta/nm)G_\perp(\mathbf{k}t) = 0$. To extract this information from the general expression of $\Omega_\perp$ [Eq. (11)], we separate $\Omega_\perp$ into a regular part $\Omega_\perp^R$ and a singular part $\Omega_\perp^s$ and compute $\Omega_\perp^s$ according to the recipe of Section 3.

4.2. The Singular Part of the Shear Viscosity

According to Eq. (22), $\delta\Omega_\perp$ splits into four parts, which we now compute. From (18)–(20) we obtain for the rhs of Eq. (22) with $i = j = t_\alpha \equiv t$

$$\langle t|\bar{\Sigma}_S^c|t\rangle = -(a_n{}^4/a_g{}^2)\Sigma_{nn}^{tt} \tag{26}$$

$$\langle t|\bar{\Sigma}_S^c|F_t\rangle = -(a_n{}^5/a_g{}^2)(\Sigma_{nl}^{tt} + \Sigma_{nt}^{tl})\langle tl|F_t\rangle - (a_n{}^5/2ma_\epsilon a_g{}^2)\Sigma_{n\epsilon}^{tt}\langle tp^2|F_t\rangle \tag{27}$$

$$\langle F_t|\bar{\Sigma}_S^c|t\rangle = -(a_n{}^5/a_g{}^2)(\Sigma_{ln}^{tt} + \Sigma_{tn}^{lt})\langle F_t|tl\rangle - (a_n{}^5/2ma_\epsilon a_g{}^2)\Sigma_{\epsilon n}^{tt}\langle F_t|tp^2\rangle \tag{28}$$

$$\begin{aligned}\langle F_t|\bar{\Sigma}_S^c|F_t\rangle = &-(a_n{}^6/a_g{}^2)\langle F_t|tl\rangle\langle tl|F_t\rangle(\Sigma_{ll}^{tt} + \Sigma_{lt}^{tl} + \Sigma_{tt}^{ll} + \Sigma_{tl}^{lt}) \\ &- (a_n{}^6/4m^2a_\epsilon{}^2a_g{}^2)\langle F_t|tp^2\rangle\langle tp^2|F_t\rangle\Sigma_{\epsilon\epsilon}^{tt} \\ &- (a_n{}^6/2ma_\epsilon a_g{}^2)\langle F_t|tl\rangle\langle tp^2|F_t\rangle(\Sigma_{l\epsilon}^{tt} + \Sigma_{t\epsilon}^{lt}) \\ &- (a_n{}^6/2ma_\epsilon a_g{}^2)\langle F_t|tp^2\rangle\langle tl|F_t\rangle(\Sigma_{\epsilon l}^{tt} + \Sigma_{\epsilon t}^{tl})\end{aligned} \tag{29}$$

In Eqs. (26)–(29), Σ_S^c, F_t, and $\Sigma_{ij}^{\alpha\beta}$ are all evaluated at the point $\mathbf{k}$, z, whereas $\alpha = l$ denotes the longitudinal polarization $[\boldsymbol{\epsilon}_l(\mathbf{k}) = \mathbf{k}/|\mathbf{k}|]$ and $\alpha = t$ one of the transverse polarizations $[\boldsymbol{\epsilon}_t(\mathbf{k})\cdot\boldsymbol{\epsilon}_l(\mathbf{k}) = 0]$. The new vectors $\langle tl|$ and $\langle tp^2|$ which appear in Eqs. (26)–(29) are respectively defined as the direct products $[u_t(p)/a_g][u_l(p)/a_g]$ and $[u_t(p)/a_g]p^2$, the a_i being the normalization constants[4] $a_i{}^2 = \langle u_i|u_i\rangle$ with $a_g \equiv a_{g_t}$ for $i = l, t_1, t_2$. The various $\Sigma_{ij}^{\alpha\beta}$ functions in (26)–(29) have been defined through Eq. (19). They have the appearance of mode–mode coupling integrals as introduced in the study of critical phenomena by, for example, Kawasaki.[24] For instance, Σ_{nn}^{tt} of Eq. (26) can be written explicitly, according to (19), as

$$\Sigma_{nn}^{tt}(\mathbf{k}, t) = -i\int \frac{d\mathbf{l}}{8\pi^3}\, [\boldsymbol{\epsilon}_t(\mathbf{k})\cdot\mathbf{l}]^2 V_l(V_l - V_{l-k})G_{nn}(\mathbf{l}, t)G_{nn}(\mathbf{k} - \mathbf{l}, t) \tag{30}$$

Our next purpose will be to evaluate these mode coupling integrals explicitly in the limit of vanishing k.

4.3. The Long-Wavelength Limit

In order to compute the long-time limit of $\eta(\mathbf{k} = 0, t)$ and hence of the hydrodynamic transport equation (25), we have to evaluate (26)–(29) up to terms of $O(k^2)$. Here we have to take into account that, because of momentum conservation, the functions $\langle F_t|tl\rangle$ and $\langle F_t|tp^2\rangle$ in Eqs. (27)–(29) are already of $O(k)$. In view of this we can rewrite Eqs. (26)–(29), keeping only terms of $O(k^2)$, as

$$\langle t|\bar{\Sigma}_S^c|t\rangle = -(a_n{}^4/a_g{}^2)ik^2I_1(z) \tag{31}$$

$$\langle t|\bar{\Sigma}_S^c|F_t\rangle = -(a_n{}^5/a_g{}^2)\langle tl|F_t(\mathbf{k}z)\rangle ikI_2(z) \tag{32}$$

$$\langle F_t|\bar{\Sigma}_S^c|t\rangle = -(a_n{}^5/a_g{}^2)\langle F_t(\mathbf{k}z)|tl\rangle ikI_2(z) \tag{33}$$

$$\begin{aligned}\langle F_t|\bar{\Sigma}_S^c|F_t\rangle = &-(a_n{}^6/a_g{}^2)\langle F_t(\mathbf{k}z)|tl\rangle\langle tl|F_t(\mathbf{k}z)\rangle iI_3(z)\\ &-(a_n{}^6/4m^2a_\epsilon{}^2a_g{}^2)\langle F_t(\mathbf{k}z)|tp^2\rangle\langle tp^2|F_t(\mathbf{k}z)\rangle iI_4(z)\end{aligned} \tag{34}$$

Where we have introduced the following four mode coupling integrals

$$\begin{aligned}I_1(t) = &-\int \frac{d\mathbf{l}}{8\pi^3}(\boldsymbol{\epsilon}_t\cdot\mathbf{l})^2V_lG_{nn}(\mathbf{l}, t)\\ &\times\left\{\left[\hat{\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{l}}G_{nn}(\mathbf{l}, t)\right]\hat{\mathbf{k}}\cdot\left(\frac{\partial}{\partial\mathbf{l}}V_l\right) + \frac{1}{2}G_{nn}(\mathbf{l}, t)\left(\hat{\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{l}}\right)^2V_l\right\}\end{aligned} \tag{35}$$

$$I_2(t) = 4\int \frac{d\mathbf{l}}{8\pi^3}(\boldsymbol{\epsilon}_t\cdot\hat{\mathbf{l}})^2(\hat{\mathbf{k}}\cdot\hat{\mathbf{l}})^2lV_l{}^2G_{nn}(\mathbf{l}, t)G_{nl}(\mathbf{l}, t) \tag{36}$$

$$\begin{aligned}I_3(t) = &\,4\int \frac{d\mathbf{l}}{8\pi^3}(\boldsymbol{\epsilon}_t\cdot\hat{\mathbf{l}})^2(\hat{\mathbf{k}}\cdot\hat{\mathbf{l}})^2l^2V_l{}^2[G_{nn}(\mathbf{l}, t)G_{ll}(\mathbf{l}, t) + G_{nl}(\mathbf{l}, t)G_{ln}(\mathbf{l}, t)]\\ &+ 2\int \frac{d\mathbf{l}}{8\pi^3}(\boldsymbol{\epsilon}_t\cdot\hat{\mathbf{l}})^2[1 - 2(\hat{\mathbf{k}}\cdot\hat{\mathbf{l}})^2]l^2V_l{}^2G_{nn}(\mathbf{l}, t)G_\perp(\mathbf{l}, t)\end{aligned} \tag{37}$$

$$I_4(t) = \int \frac{d\mathbf{l}}{8\pi^3}(\boldsymbol{\epsilon}_t\cdot\hat{\mathbf{l}})^2l^2V_l{}^2[G_{nn}(\mathbf{l}, t)G_{\epsilon\epsilon}(\mathbf{l}, t) - G_{n\epsilon}(\mathbf{l}, t)G_{\epsilon n}(\mathbf{l}, t)] \tag{38}$$

In Eqs. (35)–(38) the unit vector orthogonal to $\mathbf{k}$ has been simply denoted $\boldsymbol{\epsilon}_t \equiv \boldsymbol{\epsilon}_t(\mathbf{k})$ and hence the angular integrations can be performed immediately. As far as the $\mathbf{l}$ dependence of the various correlation functions $G_{ij}(\mathbf{l}, t)$ in (35)–(38) is concerned, we have summarized our knowledge of this in Section 2.3.

4.4. The Long-Time Limit

According to Section 2, each of the hydrodynamic correlation functions $G_{ij}(\mathbf{l}, t)$ in Eqs. (35)–(38) can be split into a hydrodynamic part $G_{ij}^H(\mathbf{l}, t)$ and a nonhydrodynamic part $\bar{G}_{ij}(\mathbf{l}, t)$. The nonhydrodynamic part $\bar{G}_{ij}$ is due to the relaxation modes, i.e., those modes whose damping rate remains finite as $\mathbf{l}$ vanishes. Hence for vanishingly small $\mathbf{l}$ and $t \sim O(l^{-2})$ we can replace the $G_{ij}(\mathbf{l}, t)$ functions of Eqs. (35)–(38) by the $G^H_{ij}(\mathbf{l}, t)$ of Eqs. (4)–(10). We will assume, moreover, that for large values of $\mathbf{l}$, $G_{ij}(\mathbf{l}, t)$ rapidly tends to zero for large t. Concretely we will approximate $I_2(t)$ [Eq. (36)], for example, as

$$I_2(t) \approx 4 \int' \frac{d\mathbf{l}}{8\pi^3} (\boldsymbol{\epsilon}_t \cdot \hat{\mathbf{l}})^2 (\hat{\mathbf{k}} \cdot \hat{\mathbf{l}})^2 l V_l^2 G^H_{nn}(\mathbf{l}, t) G^H_{nl}(\mathbf{l}, t) \tag{39}$$

where the prime indicates that the integration is restricted to values of l below some cutoff value l_0. As it stands Eq. (39), and similar ones for (35)–(38), is the second physical assumption we make in the present theory. Using the expressions given in (4)–(10), we can easily evaluate the mode coupling integrals (35)–(38). For example, for $I_2(t)$ we obtain, according to (39),

$$I_2(t) = -\frac{i}{15\pi^2} \frac{k_D}{n^2\beta^2} (\Gamma t)^{-3/2} \int_0^{l_0(\Gamma t)^{1/2}} dx\, x^2 (\exp - x^2) \sin\left(2\omega_p t + \frac{\omega_p \gamma}{2\Gamma} x^2\right) \tag{40}$$

For large values of t the upper integration limit in (40) can be pushed to infinity with an exponentially small error. If we take this limit, we obtain[25]

$$\int_0^\infty dx\, x^2 (\exp - x^2) \sin(a + bx^2) = \tfrac{1}{4}\sqrt{\pi}\, (1 + b^2)^{-3/4} \sin(a + \tfrac{3}{2} \tan^{-1} b) \tag{41}$$

and the mode coupling integrals (35)–(38) become

$$I_1(t) = k_D^{-2} \alpha^2 (\pi k_D^2 \Gamma t)^{-3/2} [1 + (1 + \delta^2)^{-3/4} \cos(2\omega_p t + \tfrac{3}{2} \tan^{-1} \delta)] \tag{42}$$

$$I_2(t) = -2ik_D^{-1} \alpha^2 (\pi k_D^2 \Gamma t)^{-3/2} (1 + \delta^2)^{-3/4} \sin(2\omega_p t + \tfrac{3}{2} \tan^{-1} \delta) \tag{43}$$

$$\begin{aligned} I_3(t) = {} & 4\alpha^2 (\pi k_D^2 \Gamma t)^{-3/2} (1 + \delta^2)^{-3/4} \cos(2\omega_p t + \tfrac{3}{2} \tan^{-1} \delta) \\ & + 6\alpha^2 [\pi k_D^2 (D_\perp + \tfrac{1}{2}\Gamma) t]^{-3/2} (1 + \delta_\perp^2)^{-3/4} \cos(\omega_p t + \tfrac{3}{2} \tan^{-1} \delta_\perp) \end{aligned} \tag{44}$$

$$\begin{aligned} I_4(t) = {} & 5\alpha^2 (c_V^0/c_V) [\pi k_D^2 (D_T + \tfrac{1}{2}\Gamma) t]^{-3/2} (1 + \delta_T^2)^{-3/4} \\ & \times \cos(\omega_p t + \tfrac{3}{2} \tan^{-1} \delta_T) \end{aligned} \tag{45}$$

where $\alpha^2 = k_D^5/120n^2\beta^2$, $\delta = \omega_p\gamma/\Gamma$, $\delta_\perp = \omega_p\gamma/(\Gamma + 2D_\perp)$, and $\delta_T = \omega_p\gamma/(\Gamma + 2D_T)$, and all other symbols have been defined previously. Besides the mode coupling integrals I_i ($i = 1,\ldots, 4$), Eqs. (31)–(34) also depend on the value of the matrix elements $\langle F_t(\mathbf{k}z)|tl\rangle$ and $\langle F_t(\mathbf{k}z)|tp^2\rangle$ and their transposed counterparts. As $F_t(\mathbf{k}z)$ only depends on the regular collision operator $\Sigma_R(\mathbf{k}z)$, we can replace $F_t(\mathbf{k}z)$ by $F_t(\mathbf{k}, z = 0)$ in Eqs. (31)–(34) as far as the leading terms of their long-time behavior is concerned. Moreover, if, as here, we are only interested in the long-time behavior of the *spatially local* shear viscosity $\eta(\mathbf{k} = 0, t)$ we need to know these matrix elements only up to $O(k)$. Comparing $\langle F_t(\mathbf{k}z)|tl\rangle$ with (24), we can write

$$\langle F_t(\mathbf{k}z)|tl\rangle = -i(k/a_n{}^2a_g)\eta_R^{TK}(\mathbf{k}z) \tag{46}$$

where η_R^{TK} is the total (= kinetic + potential)–kinetic part of the shear viscosity, i.e., the contribution of the second term of Eq. (11) with $\bar{\Sigma}^c$ deleted from the last $(\bar{\Sigma}^0 + \bar{\Sigma}^c)$ factor, calculated with the regular part of the collision operator. Finally, because of rotational invariance, the matrix element $\langle F_t(\mathbf{k}z)|tp^2\rangle$ is of $O(k^2)$ and can be neglected here.

4.5. The Asymptotic Shear Viscosity

Using the definition (24) and the results of the previous sections, we can write down our final result for the nonlocal (in time) shear viscosity $\eta(\mathbf{0}t)$. Let us introduce $\delta\eta(t) = \eta(\mathbf{0}t) - \eta_R(\mathbf{0}t)$, where $\eta_R(\mathbf{0}t)$ denotes the value of $\eta(\mathbf{0}t)$ in the absence of the mode coupling terms, i.e., its regular or exponentially decaying part. We then obtain for $\delta\eta(t)$

$$\begin{aligned}\delta\eta(t) = \frac{\lambda\omega_p}{120}\eta_0\Bigg\{&(\pi k_D{}^2\Gamma t)^{-3/2}\left[1 + (1+\delta^2)^{-3/4}\cos\left(2\omega_p t + \frac{3}{2}\tan^{-1}\delta\right)\right]\\ &- 2\,\frac{\eta_R^{KT} + \eta_R^{TK}}{\eta_0}(\pi k_D{}^2\Gamma t)^{-3/2}(1+\delta^2)^{-3/4}\sin\left(2\omega_p t + \frac{3}{2}\tan^{-1}\delta\right)\\ &- 4\,\frac{\eta_R^{KT}}{\eta_0}\frac{\eta_R^{TK}}{\eta_0}(\pi k_D{}^2\Gamma t)^{-3/2}(1+\delta^2)^{-3/4}\cos\left(2\omega_p t + \frac{3}{2}\tan^{-1}\delta\right)\\ &- 6\,\frac{\eta_R^{KT}}{\eta_0}\frac{\eta_R^{TK}}{\eta_0}\left[\pi k_D{}^2\left(D_\perp + \frac{\Gamma}{2}\right)t\right]^{-3/2}(1+\delta_\perp{}^2)^{-3/4}\\ &\times\cos\left(\omega_p t + \frac{3}{2}\tan^{-1}\delta_\perp\right)\Bigg\}\end{aligned} \tag{47}$$

where $\lambda = k_D{}^3/n$ is the plasma expansion parameter and $\eta_0 = nm\omega_p/k_D{}^2$ has the dimensions of a viscosity but is independent of the plasma parameter λ. If we compare (47) with the long-time tail of the shear viscosity of a neutral fluid,[2] we can observe three major differences: (1) Except for the very first term of (47), all remaining contributions oscillate with a frequency ω_p or $2\omega_p$; (2) the coefficients of the tail are not purely thermodynamic; (3) the different terms do not have the same overall sign. The overall $t^{-3/2}$ decay of the neutral fluids is, however, recovered here also. Finally, the detailed dependence of

(47) on λ is largely unknown but the overall amplitude of the oscillations should increase with the coupling λ, as observed for the case of the self-diffusion in Ref. 8, while the first term of (47) should be the leading contribution for small λ as assumed in Ref. 19.

Recently,[28] it has been shown that because of the Coulomb singularity the one-component plasma can show hydrodynamic behavior only in the limit of strong coupling. This could explain the appearance of the non-hydrodynamic quantity η^{KT} in our two-mode coupling result (47). The conjecture we make is that in the limit of strong coupling, higher order mode coupling contributions will come into play which eventually yield a hydrodynamic expression for $\delta\eta(t)$. However, this remains to be shown.

5. DISCUSSION

Starting from first principles we have initiated in fairly simple terms a detailed study of the long-wavelength, long-time behavior of a classical one-component plasma for arbitrary coupling constant $\lambda = k_D{}^3/n$. After recalling, in Section 2, the small-**k**, large-t expression of the hydrodynamic correlation functions $G_{ij}(\mathbf{k}t)$ (i and $j = 1,..., 5$) obtained previously,[4] we have studied the feedback effect of these hydrodynamic propagators onto themselves through their influence on the transport matrix $\Omega_{ij}(\mathbf{k}t)$, which governs the evolution of the G_{ij} through Eq. (3). The full transport matrix Ω_{ij} was separated rigorously into a regular and a singular part in Section 3. Here we have introduced our major approximation by assuming that the leading singularity of $\Omega_{ij}(\mathbf{k}z)$ or leading long-time behavior of $\Omega_{ij}(\mathbf{k}t)$ is due to the coupling of two hydrodynamic propagators. This contribution was seen to originate from the disconnected part of the nonlocal collision operator Σ_D^c of Eq. (15), which by itself represents a major improvement over the standard BGL operator. Within this general framework we have analyzed in Section 4 the long-time behavior of the nonlocal shear viscosity $\eta(\mathbf{k} = 0, t)$. An *oscillatory* long-time tail was revealed, which is displayed in Eq. (47). Although not completely unexpected here,[2] such a slow, long-time decay has some interesting consequences, which we will now briefly review.

5.1. Nonexistence of a Naive Gradient Expansion of the Hydrodynamic Equations

The nonlocal transport equation (23) together with the definition (24) can be written as a generalized or nonlocal transverse Navier–Stokes equation:

$$\partial_t G_\perp(\mathbf{k}, t) + (k^2/nm)\int_0^t d\tau\, \eta(\mathbf{k}, \tau)G_\perp(\mathbf{k}, t - \tau) = 0 \qquad (48)$$

In the limit of small gradients we can expand the nonlocal kernel as

$$\int_0^t d\tau\, \eta(\mathbf{k}, \tau)G_\perp(\mathbf{k}, t-\tau) = \sum_{n=0}^{\infty}\left\{\int_0^t d\tau\, [(-\tau)^n/n!], \eta(\mathbf{k}, \tau)\right\} \partial_t^n G_\perp(\mathbf{k}, t) \quad (49)$$

and compute $\partial_t^n G_\perp(\mathbf{k}, t)$ from Eq. (48). Since, according to Eq. (48), $\partial_t G_\perp \sim O(k^2)$, we generate in this way through the rhs of Eq. (49) a small gradient expansion of the transport equation (48). If we also take the long-time limit $\int_0^t d\tau\, \tau^n\eta(\tau) \simeq \int_0^\infty d\tau\, \tau^n\eta(\tau)$, then the first term yields exactly the usual or local transverse Navier–Stokes equation:

$$\partial_t G_\perp(\mathbf{k}, t) + (k^2/nm)\eta G_\perp(\mathbf{k}, t) = 0 \quad (50)$$

with a shear viscosity η defined as $\eta = \int_0^\infty d\tau\, \eta(\mathbf{k} = 0, \tau)$. It was generally believed that by pushing this small-k, large-t expansion further, one could generate hydrodynamic equations containing higher order gradients, i.e., the so-called Burnett $[\sim O(k^3)]$ and super-Burnett $[\sim O(k^4)]$ terms. Because of the slow decay of, for instance, $\eta(\mathbf{0}t) \sim O(t^{-3/2})$, this expansion breaks down [for example, $\int_0^\infty d\tau\, \tau\eta(\mathbf{0}\tau) = \infty$] and the super-Burnett correction to Eq. (50) does not exist. One then has to refrain from a local description and use, for example, time-dependent transport coefficients.[26] This will thus also be the case for the one-component plasma.

5.2. Nonanalyticity of the Shear Mode

Instead of looking at the hydrodynamic equation (48) obeyed by $G_\perp(\mathbf{k}, t)$, one can also look directly at the solution of Eq. (48): $G_\perp(\mathbf{k}, z) = i[z + i(k^2/nm)\eta(\mathbf{k}z)]^{-1}$. The time behavior of $G_\perp(\mathbf{k}, t)$ is then governed by the solutions of the dispersion equation:

$$z = -i(k^2/nm)\eta(\mathbf{k}z) \quad (51)$$

The shear mode can then be defined as the solution $z = z(\mathbf{k})$ of Eq. (51) that vanishes with k, $z(\mathbf{k} = 0) = 0$. For small k we recover from Eq. (51) the usual shear mode $z = -(ik^2/nm)\eta$ with $\eta = \eta(\mathbf{k} = 0, z = 0)$. It was generally believed that the small-k or Enskog expansion[2] of the shear mode would yield a result of the form $z = -i(k^2/nm)\eta(1 + k^2\eta_2 + \cdots)$, i.e., an analytic expansion in k^2 [because of rotational symmetry,[27] $\eta(\mathbf{k}z)$ only depends on k^2]. However, as has been shown recently,[2,26] the slow decay properties of $\eta(t)$ result in a nonanalytic behavior of $\eta(\mathbf{k}z)$ for small z and $\mathbf{k}$ [Eq. (47), for example, implies that $\delta\eta(z) \sim O(z^{1/2})$ for small z]. Hence, the Enskog expansion of the hydrodynamic modes will in general be nonanalytic in k or k^2 (according to the symmetry properties of the mode considered). Therefore the higher order corrections to the Navier–Stokes equation will also be nonanalytic in the gradients. Finally, since the dispersion equation (51) is also nonanalytic in z for small z, its inversion will yield cut contributions to $G_\perp(\mathbf{k}t)$. A detailed study of this wilderness of correction terms is now underway for neutral particle fluids.[26] For the plasma case we have shown[19] that the Enskog expansion of the shear mode can be written

$$z = -i(k^2/nm)\eta(1 + k\eta_1 + \cdots),$$

i.e., a nonanalytic expansion in k^2. We will come back to this problem in future work. Here we only want to point out that the reason why the singularities appear to be weaker in the plasma case compared with neutral fluids, where $z = -i(k^2/nm)\eta(1 + k^{1/2}\eta_{1/2} + \cdots)$, can be completely ascribed to the absence of sound modes in the one-component plasma.

5.3. Nonanalyticity of the Kinetic Equation

The slow decay properties of the transport matrix $\Omega_{ij}(\mathbf{k}t)$ due to the persistence of the hydrodynamic fluctuations over long time intervals are thus seen to lead to some interesting consequences at the macroscopic level. The same hydrodynamic propagators also appear in the collision operator itself, as explicitly displayed in Eq. (20). Hence the difficulties mentioned in Sections 5.1 and 5.2 are bound to manifest themselves also at the microscopic level of the kinetic equation (1). In the plasma case it is not possible to perform a small gradient expansion of the kinetic equation (1) because of the presence of a singular mean field term in (1). Let us consider, therefore, the space-integrated fluctuations $S(\mathbf{p}_1\mathbf{p}_2; t) \equiv S(\mathbf{k} = 0, t; \mathbf{p}_1\mathbf{p}_2)$, in which case Eq. (1) reduces exactly to

$$\partial_t S(\mathbf{p}_1\mathbf{p}_2; t) + \int_0^t d\tau \int d\mathbf{p}_3\, \Sigma^c(\mathbf{p}_1\mathbf{p}_3; \tau) S(\mathbf{p}_3\mathbf{p}_2; t - \tau) = 0 \tag{52}$$

with $\Sigma^c(\mathbf{p}_1\mathbf{p}_2; t) \equiv \Sigma^c(\mathbf{k} = 0, t; \mathbf{p}_1\mathbf{p}_2)$. Notice that at $\mathbf{k} = 0$ the free flow term (Σ^0) and mean field term (Σ^s) drop out exactly from Eq. (1). Equation (52) now is equivalent to the exact (linearized) kinetic equation for the non-equilibrium velocity distribution, which for small plasma coupling constants $\lambda = k_D{}^3/n$ reduces to the linearized BGL equation when the Markovian long-time limit of (52) is considered. This Markovian limit can be obtained from (52) by an expansion similar to (49):

$$\int_0^t d\tau\, \Sigma^c(\tau) S(t - \tau) = \sum_{n=0}^{\infty} \left\{ \int_0^t d\tau\, [(-\tau)^n/n!] \Sigma^c(\tau) \right\} \partial_t{}^n S(t) \tag{53}$$

where we have left out unnecessary features. It is clear from (53) that the existence of a Markovian kinetic equation in the limit of long times will depend on the existence of the moments of the collision operator,

$\int_0^t d\tau\, \tau^n \Sigma^c(\tau)$, as t tends to infinity. Clearly, if $\Sigma^c(t)$ displays the same slow, long-time decay properties as those of the transport matrix $\Omega_{ij}(\mathbf{k}, t)$, then the expansion (53) will break down somewhere in the Markovian limit $t \to \infty$. In fact it is already obvious from Eqs. (19) and (20) that the decay properties of $\Sigma^c(t)$ are similar to those of $\Omega_{ij}(t)$ and hence that the series expansion in (53) will diverge in the Markovian long-time limit. This can be phrased differently by observing that if one naively expands the Markovian kinetic equation in powers of the plasma parameter λ, the higher order terms in the series (53) are bound to appear and hence this λ expansion will also diverge. This divergence of a naive expansion of the Markovian kinetic equation will probably appear if one goes beyond the $O(\lambda^2)$ kinetic equation. This divergence then is the plasma analog of the divergence of the density expansion for neutral fluids.[2]

These, then, are in our view the major implications of the slow decay properties of the nonlocal collision operator $\Sigma^c(\mathbf{k}t)$ and nonlocal transport matrix $\Omega_{ij}(\mathbf{k}t)$ as due to the hydrodynamic transport of the fluctuations in a one-component plasma. More detailed considerations on some of the problems alluded to above are deferred to future work.

ACKNOWLEDGMENTS

We thank P. Résibois and J. P. Hansen (Paris) for commenting on the manuscript.

REFERENCES

1. B. J. Alder and T. E. Wainwright, *Phys. Rev. A* **1**:18 (1970).
2. Y. Pomeau and P. Résibois, *Physics Reports* **19**:63 (1975) and references therein.
3. M. Baus, *Phys. Lett.* **45A**:261 (1973).
4. M. Baus, *Physica* **79A**:377 (1975).
5. M. Baus, in *IUPAP Conference on Statistical Physics, Abstracts of Contributed Papers*, Akad. Kiado, Budapest (1975), p. 225.
6. J. P. Hansen, *Phys. Rev. A* **8**:3096 (1973).
7. E. L. Pollock and J. P. Hansen, *Phys. Rev. A* **8**:3110 (1973).
8. J. P. Hansen, I. R. McDonald, and E. L. Pollock, *Phys. Rev. A* **11**:1025 (1975).
9. J. P. Hansen and I. R. McDonald, *Phys. Rev. A* **11**:2111 (1975).
10. P. Vieillefosse and J. P. Hansen, *Phys. Rev. A* **12**:1106 (1975).
11. M. H. Ernst, E. H. Hauge, and J. M. J. Van Leeuwen, *Phys. Rev. A* **4**:2055 (1971).
12. Y. Pomeau, *Phys. Rev. A* **5**:2569 (1972).
13. P. C. Clemmow and J. P. Dougherty, *Electrodynamics of Particles and Plasmas*, Addison-Wesley, London (1969) and references therein.
14. D. Foster, *Phys. Rev. A* **9**:943 (1974); D. Foster and P. C. Martin, *Phys. Rev. A* **2**:1575 (1970).
15. P. Résibois, *J. Stat. Phys.* **2**:21 (1970).
16. A. Z. Akcasu and J. J. Duderstadt, *Phys. Rev.* **188**:479 (1969).

17. P. Résibois, *Physica* **59**:587 (1972).
18. H. Gould and G. F. Mazenko, *Phys. Rev. Lett.* **35**:1455 (1975).
19. M. Baus and J. Wallenborn, *Phys. Lett.* **55A**:90 (1975).
20. G. F. Mazenko, *Phys. Rev. A* **9**:360 (1974).
21. C. D. Boley, *Phys. Rev. A* **11**:328 (1975).
22. P. Résibois and Y. Pomeau, *Phys. Lett.* A**44**:97 (1973); P. Résibois, *Physica* **70**:413 (1973); P. Résibois and Y. Pomeau, *Physica* **72**:493 (1973); M. Theodosopulu and P. Résibois, *Physica* **82A**:47 (1975).
23. J. Wallenborn and M. Baus, to be published.
24. K. Kawasaki, *Ann. Phys.* (*N.Y.*) **61**:1 (1970).
25. I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, New York (1965), p. 490.
26. M. H. Ernst and J. R. Dorfman, *J. Stat. Phys.* **12**:311 (1975).
27. G. Scharf, *Helv. Phys. Acta* **42**:5 (1969).
28. M. Baus, *Phys. Rev. A*, to appear.